Dichotomy of Hofer Growth Type on the 2-Sphere

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June 2024

All work discussed was done under the supervision of my advisors, Prof. Leonid Polterovich and Prof. Lev Buhovsky (TAU), together with Prof. Egor Shelukhin (Universite de Montreal).

A symplectic manifold is a smooth manifold M with a symplectic 2-form ω. For our purpose, ω is a tool which turns smooth timed functions $H : M \times \mathbb{R} \to \mathbb{R}$, also called Hamiltonians, to a 1-parameter family of diffeomorphisms

$$\varphi_H : M \times \mathbb{R} \to M,\ \varphi_H^t(x) := \varphi_H(x,t),\ \varphi_H^0 = \mathbb{1}_M,$$

by solving the Hamiltonian flow ODE. The group of Hamiltonian Diffeomorphisms is the set of time-1 flows generated by some Hamiltonian, denoted by

$$\mathrm{Ham}(M,\omega) := \left\{\varphi_H^1 | H : M \times \mathbb{R} \to \mathbb{R}\right\}.$$

This is a group by composition. On this group, one can define the bi-invariant Hofer metric as

$$d_H(\mathbb{1},\varphi) := \inf_H \int_0^1 \|H_t\|_{L_\infty} dt,$$

where the infimum is taken over all Hamiltonians H (with some normalization condition) such that $\varphi = \varphi_H^1$. The distance between two elements $\varphi, \psi \in \mathrm{Ham}(M,\omega)$ is given by $d_H(\varphi,\psi) := d_H(\mathbb{1}, \varphi^{-1}\psi)$.

There is much to be said about this metric space $(\mathrm{Ham}(M,\omega), d_H)$, and some of its most basic properties are highly non-trivial, to the extent that even proving it is non-degenerate is an involved process which requires the development of more machinery - even the fundamental group of this space is unknown, except for a small collection of specific manifolds.

One particular unknown is the "Dichotomy of Asymptotic Growth Type Conjecture" which is the following: Take some $H \in C^\infty(M)$, Hamiltonian independent of time. One can show that for every t, $\varphi_H^t \in \mathrm{Ham}(M, \omega)$, and that the limit $\overline{d}(H) := \lim_{t\to\infty} \frac{d_H(\mathbb{1}, \varphi_H^t)}{t} \geq 0$ exists and is finite. The conjecture claims that $\overline{d}(H) = 0$ iff $d_H(\mathbb{1}, \varphi_H^t)$ is bounded, i.e. that $t \mapsto d_H(\mathbb{1}, \varphi_H^t)$ either grows linearly, or is bounded.

We prove the Dichotomy Conjecture for the case of $M = \mathbb{S}^2$ by introducing the notion of a "Symmetrization" of a Hamiltonian, which is a map $\Sigma : C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2)$ which takes some Hamiltonian, and returns a Hamiltonian which is a function of the height, symmetric about the origin. Using this tool, we managed to prove that

$$\overline{d}(H) = 0 \iff \Sigma(H) \equiv 0 \iff \forall t, \quad d_H(\mathbb{1}, \varphi_H^t) \leq 11\,\mathrm{Area}(\mathbb{S}^2).$$

Note this result is stronger than the Dichotomy Conjecture. Namely, it shows Hamiltonians with non-linear growth are universally bounded. We call this fact "Enhanced Dichotomy".

During the talk, I plan on giving a more in-depth introduction to $\mathrm{Ham}(M, \omega)$ and d_H, give a partial definition of Σ, and discuss our proof of the "Enhanced Dichotomy".